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Noise-Sustained Structure, Intermittency, and the Ginzburg–Landau Equation

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The time-dependent generalized Ginzburg–Landau equation is an equation that is related to many physical systems. Solutions of this equation in the presence of low-level external noise are studied. Numerical solutions of this equation in the *stationary* frame of reference and with a *nonzero* group velocity that is greater than a critical velocity exhibit a selective spatial amplification of noise resulting in spatially growing waves. These waves in turn result in the formation of a dynamic structure. It is found that the *microscopic* noise plays an important role in the *macroscopic* dynamics of the system. For certain parameter values the system exhibits intermittent turbulent behavior in which the random nature of the external noise plays a crucial role. A mechanism which may be responsible for the intermittent turbulence occurring in some fluid systems is suggested.

KEY WORDS: Dynamical systems; nonlinear dynamics; chaos; turbulence; intermittent turbulence; intermittency; noise; external noise; fluctuations; external fluctuations; Ginzburg–Landau equation; amplitude equation; spatially growing waves; convective instability; spatial instability; secondary instability; noise-sustained structure; noise amplification; spatial noise amplification; pattern formation.

1. INTRODUCTION

Many of the structures in nature are dynamic: the plume of smoke rising from a cigarette, the regular and chaotic patterns in the wake of a cylinder, the fluid turbulence downstream from the entrance to a pipe, and the regular and chaotic rolls in convective flows. Since noise is an element common to these and all physical systems, some natural questions which arise

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are: How important are low levels of noise in the formation and dynamics of structures in nature? To what degree is the chaotic behavior generated directly from the equations of motion and to what degree is it the result of the amplification of external noise?

Deterministic chaos has been studied extensively and found to occur in low-dimensional systems such as coupled ordinary differential equations and one-dimensional maps (e.g., see Refs. 1 and 2). It has also been found to occur in more complex systems such as partial differential equations,[3] numerical solutions of the Navier–Stokes equations,[4] and experimental fluid systems such as Taylor–Couette flow.[5] However, chaotic behavior resulting from the amplification of noise in comparison has received little attention.

One type of system in which noise can be amplified is that in which local instabilities exist in phase space.[1,6,7] For example, if a system has a periodic orbit which is stable overall but has local regions of instability, external noise will be amplified in the unstable regions. Thus an orbit which would be periodic in the absence of noise will be aperiodic and irregular in the presence of noise, assuming that the noise level and region of instability are sufficiently large.

In this paper we study systems in which a different type of mechanism—*spatial* noise amplification—is responsible for the amplification of the noise. In this type of system small fluctuations grow as they move spatially. Therefore this behavior can only occur in systems that have spatial extent, such as partial differential equations and large numbers of coupled maps. Although the concept of spatially growing waves has been around for a while,[8–10] the importance of external fluctuations in such systems does not appear to be generally recognized, possibly because of the complexity of the systems usually considered. A primary purpose of the present paper is to show that spatially growing waves occur in simple systems that can be easily solved numerically, thus making this mechanism of noise amplification more accessible to study. Also we wish to show that it is of fundamental importance to include external fluctuations in the modeling of some systems, to show that low levels of external noise can play a major role in the macroscopic dynamics of systems, and to show that noise-sustained structure, a notion introduced in Ref. 11, also occurs in solutions of partial differential equations. A noise-sustained structure is a structure that is sustained by the presence of microscopic noise and thus owes its existence to the presence of the noise.

In Ref. 11 the effects of low-level external noise on a difference equation consisting of 200 points coupled with logistic maps was studied. After transients had settled down this system exhibited a spatial exponential growth of fluctuations resulting in a complex chaotic structure, the structure being maintained by the presence of the fluctuations. Thus in this system low-level fluctuations played a fundamental role.

Here we study the effects of external fluctuations on solutions of a model partial differential equation which appears in many areas[3,12–16]—the time-dependent generalized Ginzburg–Landau equation:

$$\frac{\partial \Psi}{\partial t} = a\Psi + b\frac{\partial^2 \Psi}{\partial X^2} - c\,|\Psi|^2\,\Psi \tag{1}$$

where the dependent variable Ψ is in general complex; a, b, and c are constants which are in general complex; $b_r > 0$; and $X = x - vt$ is a coordinate in a frame of reference moving at the group velocity v. Real and imaginary parts of complex quantities are subscripted with r and i, respectively. Equation (1) in the (X, t) variables has been numerically studied by a number of researchers.[3,17–20] As will be seen shortly, in order for noise to have an important effect the equation must be solved in the stationary frame of reference [i.e., in the (x, t) variables]. Transforming to the stationary frame gives

$$\frac{\partial \psi}{\partial t} = a\psi - v\frac{\partial \psi}{\partial x} + b\frac{\partial^2 \psi}{\partial x^2} - c\,|\psi|^2\,\psi \tag{2}$$

where ψ is the transformed variable. We find that this equation, in the presence of low-level external noise and with a nonzero group velocity which is greater than a critical value, exhibits behavior similar to that of Ref. 11 (i.e., fluctuations growing spatially to macroscopic proportion resulting in a dynamic structure). Thus we are mainly interested in systems for which this equation has a nonzero group velocity, such as plane Poiseuille flow[14] and wind-induced water waves.[15]

2. LINEAR STABILITY

Let us first inquire into the linear stability of the equilibrium solution $\psi = 0$ of Eq. (2). Therefore we consider the equation

$$\frac{\partial \psi}{\partial t} = a\psi - v\frac{\partial \psi}{\partial x} + b\frac{\partial^2 \psi}{\partial x^2} \tag{3}$$

Let $\psi_0(x) = \psi(x, 0)$ be a small initial localized perturbation about the equilibrium state $\psi = 0$. As shown in Refs. 21 and 22 a perturbation can undergo three basic types of behavior. We find that these three different types of behavior occur for solutions of Eq. (3). In the following definitions the boundaries are taken to be at $x = -\infty$ and $x = \infty$. The first type of behavior is defined by

$$\lim_{t \to \infty} |\psi(x, t)| \to \infty \tag{4}$$

for an arbitrary fixed value of x. Condition (4) corresponds to the system being *absolutely unstable*. If this condition is satisfied the perturbation is growing and spreading such that its edges are moving in opposite directions. This is the type of instability with which most of us are familiar. The second type of behavior is defined by

$$\lim_{t \to \infty} |\psi(X' + v't, t)| \to 0 \tag{5}$$

for any v' and for an arbitrary fixed value of X'. Condition (5) corresponds to the system being *absolutely stable*. If this condition is satisfied the perturbation is damped in any frame of reference. The third type of behavior is defined by

$$\lim_{t \to \infty} |\psi(x, t)| \to 0 \qquad \text{and} \qquad \lim_{t \to \infty} |\psi(X' + v't, t)| \to \infty \tag{6}$$

for some v' and for arbitrary fixed values of x and X', respectively. Condition (6) corresponds to the system being *spatially unstable*. If this condition is satisfied the perturbation is damped (as $t \to \infty$) at any given stationary point, but a moving frame of reference may be found in which the perturbation is growing. Therefore, even though the perturbation is growing and spreading, both edges of the perturbation are moving in the same direction, thus allowing the system behind the perturbation to return to its undisturbed state. In the plasma physics and fluid mechanics literature[(21–23)] this type of instability is referred to as a convective instability. Here we use the term spatial since the term convective is usually associated with fluids. This is the type of instability in which we are mainly interested in this paper since it will result, as also noted by Ref. 22, in the amplification of noise. For the cases considered in this paper the "any v'" of condition (5) and the "some v'" of condition (6) may be replaced by "$v' = v$", where v is the group velocity of the perturbation.

Taking the boundaries at $x = -\infty$ and $x = \infty$, the subsequent evolution of the perturbation for the linear Eq. (3) is given by

$$\psi(x, t) = \frac{e^{at}}{2(\pi bt)^{1/2}} \int_{-\infty}^{\infty} dx' \, \psi_0(x') \exp\left[-\frac{(x - vt - x')^2}{4bt} \right] \tag{7}$$

This comes from simply writing ψ as a Fourier integral and performing the integration over wave number. Applying the conditions (4)–(6) to Eq. (7) we find that the equilibrium solution $\psi = 0$ of Eq. (3) is absolutely unstable if

$$a_r - \frac{v^2 b_r}{4\,|b|^2} > 0 \tag{8}$$

absolutely stable if

$$a_r < 0 \tag{9}$$

and spatially unstable if

$$a_r - \frac{v^2 b_r}{4\,|b|^2} < 0 \qquad \text{and} \qquad a_r > 0 \tag{10}$$

The first part of condition (10) may be written as $|v| > 2\,|b|\,(a_r/b_r)^{1/2}$ which simply says that the magnitude of the group velocity of the perturbation must be greater than the magnitude of the velocity at which it spreads (i.e., speed of an edge relative to the comoving frame). This critical velocity is the same as that given by the Dee–Langer marginal stability condition,[18,24] which gives the velocity of a pattern front. Note that, since $b_r > 0$, $a_r > 0$ if condition (8) is satisfied and $a_r - (v^2 b_r/4\,|b|^2) < 0$ if condition (9) is satisfied. It may be instructive to note that for an initial Gaussian perturbation, $\psi_0(x) = Ae^{-\alpha x^2}$,

$$\psi(x, t) = A\,\frac{e^{at}}{(1 + 4\alpha bt)^{1/2}}\exp[-\alpha(x - vt)^2/(1 + 4\alpha bt)] \tag{11}$$

The three types of behavior become particularly clear in examining this solution.

To see if other types of boundary conditions may have an effect on the stability conditions, let us now take the boundaries at $x = 0$ and $x = \infty$ and take $\psi(x = 0, t) = 0$ instead of taking the boundaries at $x = -\infty$ and $x = \infty$. The solution of Eq. (3) is then

$$\psi(x, t) = \int_0^\infty dk\, A(k)\, e^{\alpha(k)t} e^{(v/2b)x} \sin(kx) \tag{12}$$

where

$$A(k) = \frac{2}{\pi}\int_0^\infty dx'\,\psi_0(x')\, e^{-(v/2b)x'} \sin(kx')$$

and

$$\alpha(k) = a - \frac{v^2}{4b} - k^2 b$$

The integral over k can not be done in closed form for this case. Therefore to determine the asymptotic behavior of the integral and to apply conditions (4)–(6) we deform the contour of integration off the real axis and into the complex plane. We then use the method of steepest descent[25] which tells us that the asymptotic behavior of the integral will be determined by the behavior near the saddle point. Although the left boundary is not at $x = -\infty$, the definitions (4)–(6) will still apply if x, v', and v are restricted to be >0. For a fixed value of x the saddle point is determined by $d\alpha/dk = 0$ (k complex) which gives us $k = 0$. Thus if $\mathrm{Re}[\alpha(0)] > 0$ the system will be absolutely unstable and if $\mathrm{Re}[\alpha(0)] < 0$ the system will be either absolutely stable or spatially unstable. These results are identical with the previous results when the left boundary was at $x = -\infty$. To distinguish between the system being absolutely stable and spatially unstable we look at the asymptotic behavior of the integral in a frame of reference moving at velocity v' [i.e., replace x in Eq. (12) by $X' + v't$ where X' is fixed] and assume that $v' > 0$ and $v > 0$. Writing the sine function as a sum of two exponentials gives us $\gamma(k) = a - (v^2/4b) - bk^2 + (vv'/2b) \pm ikv'$ for the coefficient of t in the exponentials. The saddle point is determined by $d\gamma/dk = 0$ giving $k = \pm(iv'/2b)$. Putting this value of k back into γ gives $\gamma = a - [(v - v')^2/4b]$. The maximum value of γ_r occurs when $v' = v$. Thus the system will be absolutely stable if $a_r < 0$ and spatially unstable if both $a_r > 0$ and $\mathrm{Re}[\alpha(0)] < 0$. We therefore find that the stability conditions (8)–(10) are left unchanged as one may have expected.

Up to this point the system has been unbounded. Let us now take the boundaries at $x = 0$ and $x = L$ (with $\psi(0, t) = 0$) and assume that $v > 0$ and that the initial perturbation is near the boundary $x = 0$. In order for the notion of spatial instability to be a meaningful concept it is necessary that the dimensions of the system be large enough so that a perturbation grows significantly before leaving the boundaries of the system. If we assume open boundary conditions (i.e., just as if there were no boundary) at $x = L$ the above stability conditions (8)–(10) will still apply. If boundary conditions such as $\psi = 0$ or $d\psi/dx = 0$ are taken at $x = L$ we would expect that for sufficiently large L the boundaries would have little effect. To test this hypotheses the boundary conditions will now be taken as $\psi(0, t) = 0$ and $\psi(L, t) = 0$. The solution of Eq. (3) is then

$$\psi(x, t) = \sum_{k=1}^{\infty} A_k e^{\alpha_k t} e^{(v/2b)x} \sin\left(\frac{\pi k}{L} x\right) \tag{13}$$

where

$$A_k = \frac{2}{L} \int_0^L dx' \, \psi_0(x') \, e^{-(v/2b)x'} \sin\left(\frac{\pi k}{L} x'\right)$$

and

$$\alpha_k = a - \frac{v^2}{4b} - \frac{\pi^2 k^2 b}{L^2}$$

Since the most rapidly growing (or least damped) mode corresponds to $k=1$, the system will be absolutely unstable if $Re[\alpha_1] > 0$ and either absolutely stable or spatially unstable if $Re[\alpha_1] < 0$. This result differs from the previous result for open boundary conditions only by order $1/L^2$. We also see that these boundary conditions have a slight stabilizing effect as one would expect. Note that the α_k are solutions to the eigenvalue equation $a\phi - v\phi' + b\phi'' = \alpha\phi$, where a prime denotes a spatial derivative.

For the boundary conditions $\psi|_{x=0} = 0$ and $(\partial^n\psi/\partial x^n)|_{x=L} = 0$ $(v > 0, n > 0)$ it is straightforward to show that the eigenvalue with the largest real part is given to order $1/L^3$ by

$$\alpha_1 = a - \frac{v^2}{4b} - \frac{\pi^2 b}{L^2} + \frac{n\pi^2 b^2}{vL^3}$$

The main point here is that boundary conditions of this type have little effect on the stability conditions if L is sufficiently large.[3]

However for periodic boundary conditions the stability conditions are drastically changed. This becomes clear by noting that if the boundary conditions are periodic, a spatially growing perturbation which would have otherwise traveled out through a boundary will be fed back through the other boundary, thus making a system which would have otherwise been spatially unstable absolutely unstable. The solution of Eq. (3) for periodic boundary conditions is

$$\psi(x, t) = \sum_{k=-\infty}^{\infty} A_k e^{\alpha_k t} e^{(2\pi i k/L)x} \tag{14}$$

where

$$A_k = \frac{1}{L}\int_0^L dx'\, \psi_0(x')\, e^{-(2\pi i k/L)x'}$$

and

$$\alpha_k = a - \frac{2\pi i k}{L} v - \frac{4\pi^2 k^2}{L^2} b$$

[3] For the boundary conditions $(\partial^m\psi/\partial x^m)|_{x=0} = 0$ and $(\partial^n\psi/\partial x^n)|_{x=L} = 0$ where $m > 0$ and $n > 0$ the eigenvalue with the largest real part is $\alpha_0 = a$ (corresponding to a constant solution to the eigenvalue equation). With these boundary conditions the system can only be absolutely stable $(a_r < 0)$ or absolutely unstable $(a_r > 0)$ and the notion of spatial instability does not apply.

Therefore for periodic boundary conditions the system is absolutely stable if $Re[\alpha_0]$ or $a_r < 0$ and absolutely unstable if $Re[\alpha_0]$ or $a_r > 0$. There are no parameter values for which the system is spatially unstable and the notion of spatial instability does not apply.

A method that is often used to establish stability criteria is the calculation of eigenvalues of the Jacobian matrix. Thus it is of interest to calculate the eigenvalues of the Jacobian matrix for the spatially discretized equation (3). We take the boundary conditions $\psi(0, t) = \psi(L, t) = 0$ since the eigenvalues can be found in closed form. Writing the spatial derivatives in Eq. (3) as second order differences (i.e.,

$$\left.\frac{\partial\psi}{\partial x}\right|_{x=x_i} = \frac{\psi(x_{i+1}, t) - \psi(x_{i-1}, t)}{2\Delta x}$$

and

$$\left.\frac{\partial^2\psi}{\partial x^2}\right|_{x=x_i} = \frac{\psi(x_{i+1}, t) + \psi(x_{i-1}, t) - 2\psi(x_i, t)}{\Delta x^2},$$

$i = 1, 2,..., N$) gives a set of N coupled ordinary differential equations with a corresponding Jacobian matrix that is tridiagonal. The eigenvalues of the tridiagonal matrix $\{f, g, h\}$ are $\lambda_s = g + 2(fh)^{1/2} \cos[s\pi/(N+1)]$, $s = 1, 2,..., N$.[26,27] Here g are the diagonal elements and f and h are the elements below and above the diagonal, respectively. We thus get for the eigenvalues of the spatially discretized Eq. (3)

$$\lambda_s = a + \frac{2b}{\Delta x^2}\left\{\left[1 - \left(\frac{v}{2b}\Delta x\right)^2\right]^{1/2} \cos\left(\frac{s\pi}{N+1}\right) - 1\right\}, \qquad s = 1, 2,..., N \quad (15)$$

In the limit as $\Delta x \to 0$ such that $L = (N+1)\,\Delta x$ for fixed L the eigenvalue with the largest real part is $\lambda = a - (v^2/4b) - (\pi^2 b/L^2)$. Note that λ is equal to α_1 in Eq. (13). For $\lambda_r > 0$ the system will be absolutely unstable. For $\lambda_r < 0$ the system will be either absolutely stable or spatially unstable. Thus an eigenvalue calculation does not distinguish between absolutely stable and spatially unstable systems since it tells us nothing about the behavior of a perturbation in the comoving frame.

If periodic boundary conditions are imposed the tridiagonal matrix $\{f, g, h\}$ will have the element f added to the upper right corner and the element h added to the lower left corner owing to the boundaries being coupled together [i.e., $\psi(x_N, t)$ will depend on $\psi(x_1, t)$ and vice versa]. The eigenvalues of this matrix are $\lambda_s = g + fe^{-2\pi is/N} + he^{2\pi is/N}$, $s = 0, 1,..., N-1$. These eigenvalues are easily found by generalizing the derivation in Ref. 28 to the case of unsymmetric matrices. Thus the eigenvalues for the spatially discretized Eq. (3) with periodic boundary conditions are

$$\lambda_s = a + \frac{2b}{\Delta x^2}\left[\cos\left(\frac{2\pi s}{N}\right) - 1\right] - \frac{iv}{\Delta x}\sin\left(\frac{2\pi s}{N}\right), \qquad s = 0, 1, \ldots, N-1 \tag{16}$$

In the limit as $\Delta x \to 0$ such that $L = (N+1)\,\Delta x$ for fixed L the eigenvalue with the largest real part is $\lambda = a$. Requiring that $\lambda_r > 0$ gives the second part of condition (10). This suggests a method for distinguishing between absolutely stable and spatially unstable systems with Jacobian eigenvalue calculations. If $\lambda_r < 0$ for a system of equations with a given set of boundary conditions and $\lambda_r > 0$ for the same system of equations with periodic boundary conditions imposed instead then, assuming that the dimensions of the system are large enough so that the given boundaries have an overall negligible effect on the behavior of the perturbation, the system will be spatially unstable.

3. NOISE-SUSTAINED STRUCTURE

As previously noted, Eq. (2) with $v = 0$ [or equivalently Eq. (1) in the (X, t) variables] has been numerically studied by a number of researchers. If $a_r > 0$ this system is absolutely unstable. If the equilibrium solution $\psi = 0$ is given a small (microscopic) local perturbation for $a_r > 0$, the perturbation will grow with time eventually reaching macroscopic proportion. If $c_r > 0$ the amplitude will saturate and thus the perturbation will grow to a finite size producing a structure, which will most likely be changing with time (i.e., dynamic). Periodic, quasiperiodic, and chaotic structures have been observed. The point here is that a single perturbation is sufficient to produce a structure for all time.

In contrast, if v is nonzero and sufficiently large, the perturbation and resultant structure will move spatially such that the structure will eventually leave the boundaries of the system. Thus the system will return to the equilibrium state. A single perturbation will therefore produce only a temporary structure. However if the system is continuously perturbed by microscopic fluctuations the system will be unable to return to the equilibrium state and a new state will be established which is sustained by the presence of the fluctuations. We refer to such a state as a *noise-sustained state* and the corresponding structure as a *noise-sustained structure*. We assume that the spatial growth factor of the fluctuations and the length of the system are large enough such that the fluctuations will reach macroscopic proportion and produce a structure before they leave the boundaries of the system. From the previous discussion on linear stability it is clear that a necessary condition for noise-sustained structure is that the system be spatially unstable. For example, if v is sufficiently large and small fluctuations are introduced near the left boundary, the fluctuations will grow spatially until nonlinear effects enter which will cause the fluctuations to saturate ($c_r > 0$) forming a structure.

4. NUMERICAL SOLUTIONS AND DISCUSSION

We now study the effect of low-level fluctuations on numerical solutions of Eq. (2). Second-order Runge–Kutta is used in the time differencing and fourth-order differencing is used in the space differencing[(26)] except at the grid points adjacent to the boundaries where second-order differencing is used. The distance between spatial grid points is $\Delta x = 0.3$, and the time step is $\Delta t = 0.01$. The boundary conditions are $\psi(0, t) = 0$ and $\psi''(L, t) = 0$,[4] where a prime denotes a spatial derivative. Fluctuations are introduced into the system by adding, at each time step, random numbers uniformly distributed between $-r$ and r to ψ_r and ψ_i at all grid points except the left boundary.[5] Cray single precision (14 digit accuracy) is used in the calculations. We start with the initial equilibrium state $\psi = 0$ and allow the system to evolve in the presence of these slight fluctuations to a state for which transients have settled down (i.e., to a statistically steady state).

Figures 1 and 2 show plots of the real and imaginary parts of ψ as a function of x for a given value of t for a few different parameter values. We see that the fluctuations near the left boundary grow spatially to macroscopic proportion resulting in the observed structure. The structures in Figs. 1 and 2 are noise-sustained structures. If the external fluctuations are removed, the structure moves out through the right boundary and the system returns to the state $\psi = 0$ everywhere, except for some slight ($< 10^{-100}$) fluctuations due to computer roundoff. Condition (10) is satisfied since $v = 6$ is greater than the critical velocity $2\,|b|\,(a_r/b_r)^{1/2}$ which equals 4 and 5.03 for the parameter values of Figs. 1 and 2, respectively.

For fixed parameter values the position at which the fluctuations are just large enough to be seen will depend on the fluctuation level r, shifting to the left/right with a larger/smaller fluctuation level. This follows from the fact that if the fluctuations are larger at their source they will show themselves at a point closer to the source. This behavior can be seen in compar-

[4] This boundary condition was chosen mainly for aesthetic reasons, since it approximates an open boundary. If other boundary conditions such as $\psi = 0$ are used at $x = L$ the behavior changes only near that boundary.

[5] The reason the noise is added at all points is that this is in some sense more physical. If instead the noise is added at only a single grid point near the left boundary the basic qualitative behavior of the system remains unchanged.

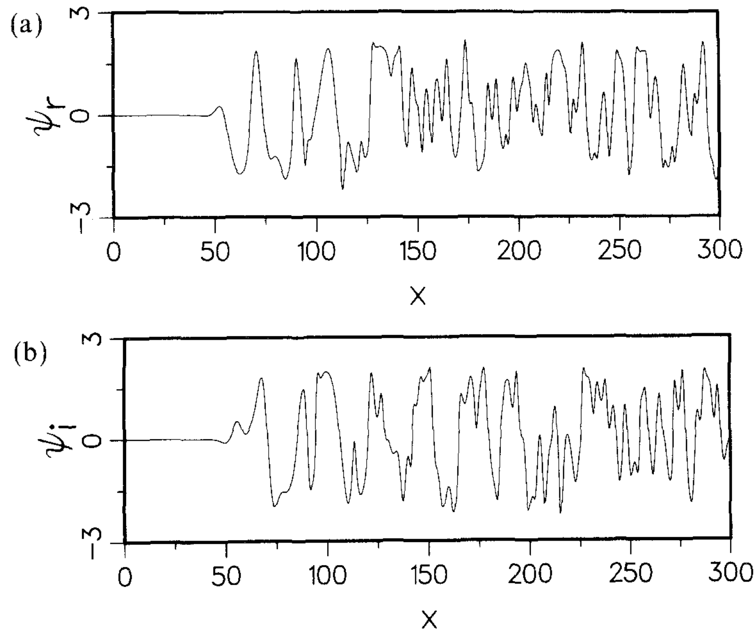

Fig. 1. Plots of ψ_r and ψ_i as a function of x for a given t ($t = 200$) after transients have settled down. $a = 2$, $v = 6$, $b_r = 1$, $b_i = -1$, $c_r = 0.5$, and $c_i = 1$. Noise level $= r = 10^{-9}$. The microscopic noise near the left boundary grows spatially to macroscopic proportion resulting in the observed structure.

ing Fig. 3 to Fig. 1a which have the same parameter values but different noise levels.

As seen in Fig. 2 even very regular structure (i.e., the sinusoidal-like pattern) may be supported by random fluctuations. This is possible since a relatively narrow band of wave numbers in the fluctuations is amplified as seen in Fig. 4, which shows time series at a point where the fluctuations are small enough to be considered linear. An example of this type of behavior in fluid systems is the existence of Tollmien–Schlichting waves,[29,30] which result from the selective amplification of random background fluctuations.

The random nature of the fluctuations plays a major role in the dynamics of the system. Thus the dynamics is not determined solely by the equations of motion [i.e. Eq. (2)] and the chaotic behavior is not associated with a strange attractor. The importance of the randomness of the fluctuations can be seen by perturbing the system sinusoidally instead of randomly. This is done by perturbing the left boundary in the following fashion: $\psi_r(0, t) = A \sin(\omega t)$ and $\psi_i(0, t) = A \cos(\omega t)$. Figure 5 shows plots of ψ_r as a function of x for the same parameter values as those in Figs. 1 and 2. The value of A is 10^{-8}. The values of ω used are those values that

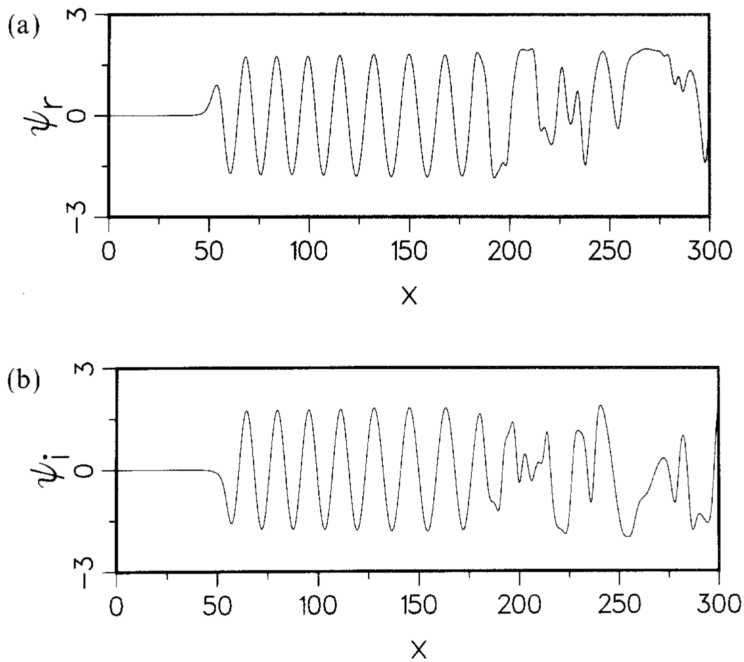

Fig. 2. Plots of ψ_r and ψ_i as a function of x for a given t ($t = 300$) after transients have settled down. $a = 2$, $v = 6$, $b_r = 2.8$, $b_i = -1$, $c_r = 0.5$, $c_i = 1$, $r = 10^{-9}$. The parameter values and noise level are identical to those of Fig. 1 except for the value of b_r.

give the largest spatial growth (to be derived later). Since the structures in Fig. 5 are very regular owing to the regular nature of the perturbations, it is clear that the random nature of the fluctuations play a major role in the dynamics of the systems with external noise.

We inquire into the stability of the structures of Fig. 5 by adding small random fluctuations to ψ_r and ψ_i at a point near the left of the structures (i.e., at $x = 75$) and allow the system to evolve to a state in which transients have settled down. These random fluctuations are in addition to the sinusoidal perturbation. Figure 6 shows the result. We see that the small fluctuations at $x = 75$ grow spatially causing the structure to become irregular at some point. If larger/smaller fluctuations are added the point where the structure becomes irregular shifts to the left/right. Thus the structures themselves are *spatially* unstable.

Figure 7 shows a time series at the point $x = 150$ for the same parameter values as those of Fig. 1a. At this point the fluctuations are well into the nonlinear region. Comparing Fig. 7 with Fig. 4a it is seen that the nonlinearity causes a breakup of the wave present in the linear region into higher-frequency components in the nonlinear region as a result of the

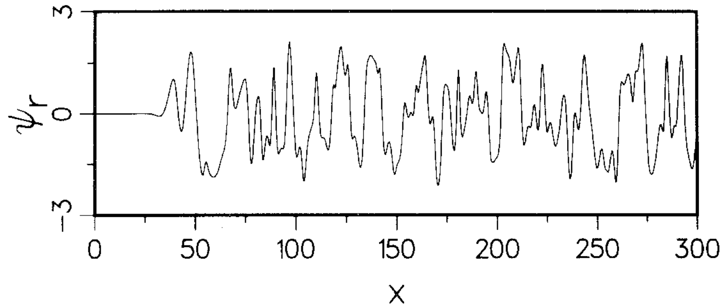

Fig. 3. Plot of ψ_r as a function of x for a given t ($t = 200$) after transients have settled down. $a = 2$, $v = 6$, $b_r = 1$, $b_i = -1$, $c_r = 0.5$, $c_i = 1$, $r = 10^{-6}$. The larger noise level causes the fluctuations to be seen at a smaller value of x (compare with Fig. 1a).

spatial instability considered in the last paragraph. This type of behavior is familiar in fluid mechanics where large eddies break up into smaller eddies.

The importance of the fluctuations are particularly apparent in the behavior of the system of Fig. 2. The spatial extent of the regular sinusoidal-like pattern in Fig. 2 varies in a random fashion with time, at times there being as many as ten or more regular waves and at other times just a few. This intermittent behavior is seen in Fig. 8, where ψ_r is plotted as a function of time for different fluctuation levels and at different spatial points. High- and low-frequency signals are seen to be interspersed. Comparing Fig. 8b to Fig. 8a we see that the signal is high frequency a larger fraction of the time at a larger value of x for a given fluctuation level. Comparing Fig. 8c with Fig. 8a we also see that the signal is high frequency a larger fraction of the time for higher fluctuation levels for a given distance from the point at which the fluctuations become macroscopic. The latter is due to the signal being more irregular at this point for larger noise levels, since the noise has not been filtered as thoroughly. To see that the distance from this point is about the same for Figs. 8a and 8c one may note that the amplitude of the signal is about the same in Figs. 4b and 4c and that the distance between the spatial points at which the signal is measured in Figs. 8a and 4b and in Figs. 8c and 4c is the same.

This intermittent behavior may be understood in the following manner: As noted previously, when the system is perturbed sinusoidally a regular structure results which is spatially unstable.[6] Thus if the spatially growing waves are regular in the linear region the structure will be regular. If there is an irregularity (i.e. a variation of the frequency, amplitude, and/or phase) in the spatially growing waves in the linear region, the

[6] The instability of periodic wave solutions is often referred to as a Benjamin-Feir instability,[31] although a better term may be an Eckhaus-Benjamin-Feir instability.[32,33]

regular structure will be disrupted and become irregular at some spatial point due to the spatial instability. If this irregularity in the spatially growing waves changes with time, as it clearly does in these numerical studies, the point where the structure becomes irregular will change with time resulting in intermittency. Notice that the noise is playing two very different roles in this intermittency mechanism. First, the noise is the source of

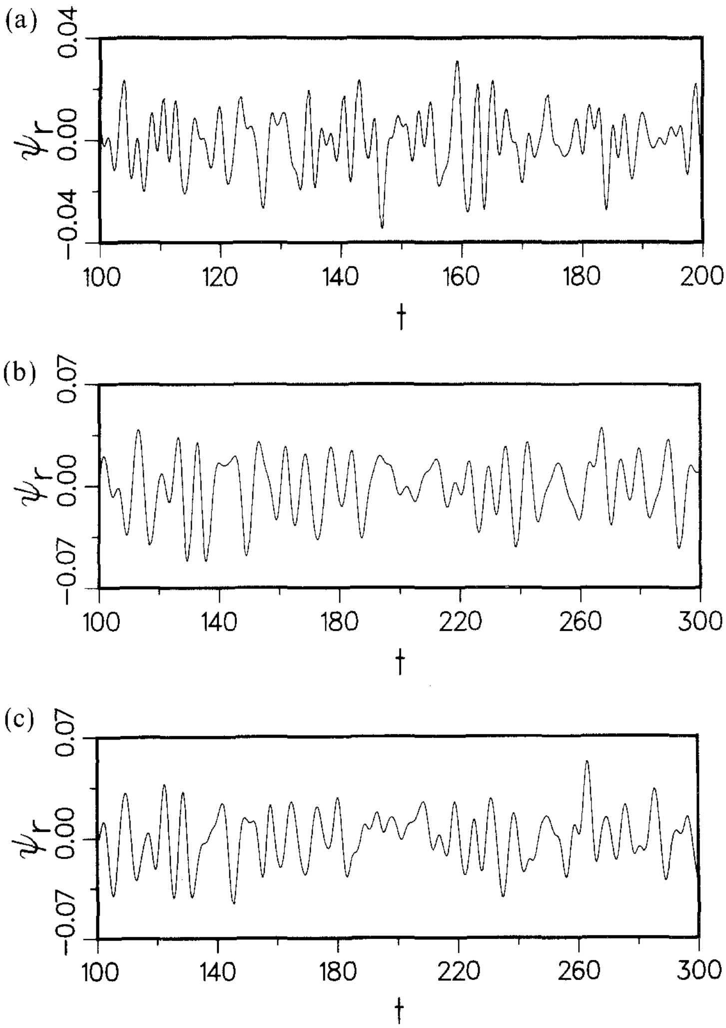

Fig. 4. Plots of ψ_r as a function of t. $a=2$, $v=6$, $b_i=-1$, $c_r=0.5$, and $c_i=1$. (a) $b_r=1$, $x=45$, $r=10^{-9}$; (b) $b_r=2.8$, $x=42$, $r=10^{-9}$, c) $b_r=2.8$, $x=25.5$, $r=10^{-6}$. At these spatial points the fluctuations are still small enough to be considered linear. It is seen that only a narrow frequency band of the original broad band noise has been amplified.

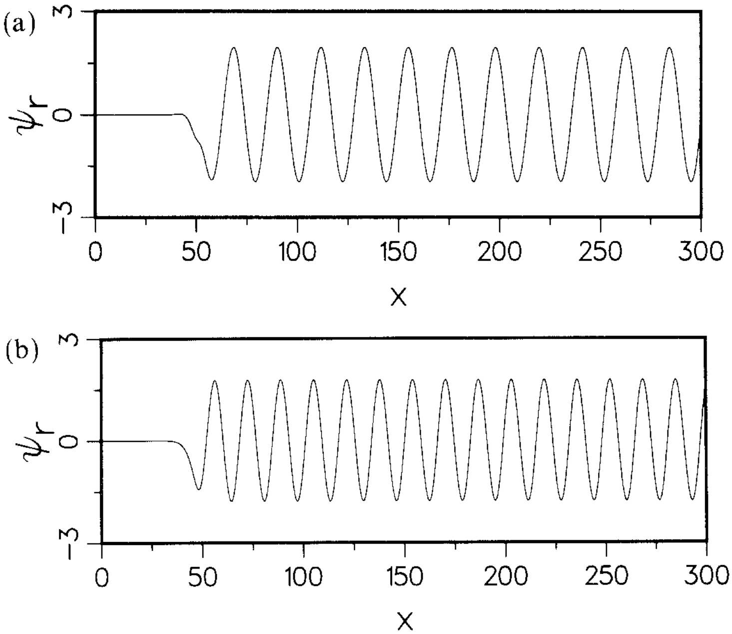

Fig. 5. Plots of ψ_r as a function of x with a sinusoidal perturbation of frequency ω and amplitude 10^{-8} at the left boundary. $a=2$, $v=6$, $b_i=-1$, $c_r=0.5$, and $c_i=1$. (a) $b_r=1$, $\omega=2$; (b) $b_r=2.8$, $\omega=0.7143$. The structures are seen to be regular as a result of the regular perturbation (compare with Figs. 1 and 2 where the perturbations are random).

the spatially growing waves and therefore the source of the regular structure. This is possible, as noted previously, since only a narrow frequency band of the original broad band noise is selectively amplified. Thus the noise, as it grows spatially, eventually forms fairly regular waves (as seen in Fig. 4) which in turn form the regular structure (as seen in Fig. 2). Second, the noise has a destabilizing effect on the very structure it produced. This is due to the fact that the spatially growing waves, having been produced by the random noise, are not completely regular but have a random component. This random component causes the regular structure, since it is spatially unstable, to be disrupted and become irregular at some point (as seen in Fig. 2).[7]

[7] The high frequency signal seen in Fig. 8 is generated by the nonlinear dynamics (as a result of the secondary instability) and is not the result of high frequency components of the noise being linearly amplified. This is supported by the fact that the highest frequencies seen in these figures would be damped in the linear region and the high frequencies exist even if periodic boundary conditions are instead imposed. Also, the same phenomenon (except that the intermittency occurs at regular intervals) may be seen by perturbing the system sinusoidally as in Fig. 5b (with $A=.01$), but in addition introducing an irregularity by

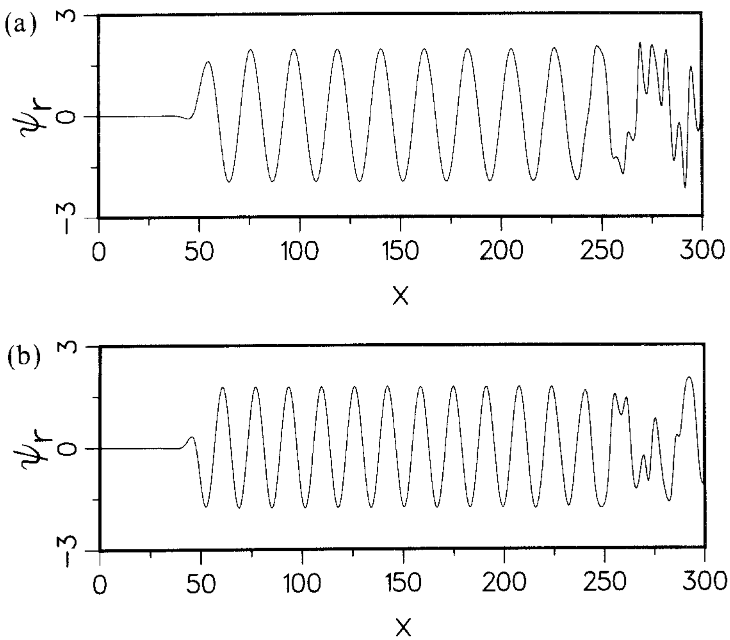

Fig. 6. Plots of ψ_r as a function of x with a sinusoidal perturbation of amplitude 10^{-8} at the left boundary. A small random number uniformly distributed between $-r_2$ and r_2 is added to ψ_r and ψ_i at $x=75$ at each time step. $a=2$, $v=6$, $b_i=-1$, $c_r=0.5$ and $c_i=1$. (a) $b_r=1$, $r_2=4(10^{-5})$, $\omega=2$; (b) $b_r=2.8$, $r_2=4(10^{-3})$, $\omega=0.7143$. The random fluctuations cause the regular sinusoidal pattern to break up at some point.

We are therefore lead to suggest the following mechanism which may be responsible for the intermittent turbulence occurring in some fluid systems. Microscopic background noise is spatially and selectively amplified resulting in the formation of spatially growing waves. When the amplitude of these waves gets large enough nonlinear effects enter and the waves become *spatially* unstable. This spatial instability causes the waves to break up into smaller wavelength components and become turbulent at some spatial point. Since the source of the waves is random noise, the irregularities in the waves will change with time and thus the point at which the pattern becomes turbulent will change with time. The result will be intermittent turbulence.

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changing the amplitude and/or frequency of the sinusoidal perturbation for one cycle every six cycles for example. In this case a "burst" is associated with the temporary change in amplitude and/or frequency of the perturbation. [Note that this sort of behavior can be seen in Fig. 4b (i.e. stretches of fairly constant frequency interrupted by sudden changes in frequency—the longer and more regular a "stretch" the more extended the regular structure).]

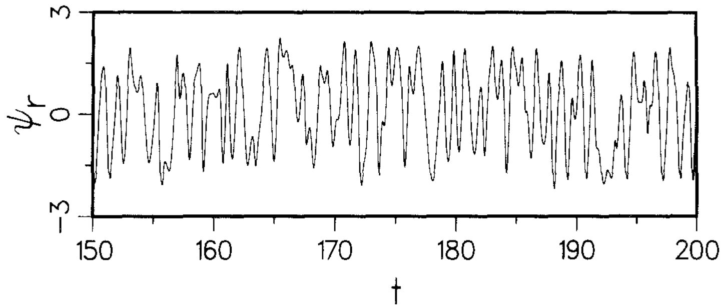

Fig. 7. Plots of ψ_r as a function of t at $x = 150$. $a = 2$, $v = 6$, $b_r = 1$, $b_i = -1$, $c_r = 0.5$, and $c_i = 1$. As a result of the nonlinearity, the frequencies seen here are much larger than those in the linear region (compare with Fig. 4a noting that the time scales differ).

A movie has been made which shows how $\psi(x, t)$ plotted as a function of x changes with time. In the movie this intermittent behavior is clearly seen. At least on the surface the behavior is very similar to the intermittent behavior which results from fluid flow over a flat plate.[30,34] In this system there are also distinct spatial regions where different types of behavior occur: a laminar region, a region of fairly regular waves, a transition region where intermittency occurs, and a turbulent region. In the transition region the fluid is turbulent a larger fraction of the time at a point further downstream from the leading edge of the plate. Also the average distance between the leading edge of the plate and the point at which the fluid becomes turbulent decreases with an increasing background fluctuation level. Both these behaviors are very similar to the behavior of the system studied in this paper.

As previously discussed the behavior of the solution will be very different if periodic boundary conditions are imposed instead. Just for comparison Fig. 9 shows plots with periodic boundary conditions imposed. We find that the overall macroscopic behavior of the solution is insensitive to low levels of external noise which is very different from the results for nonperiodic boundary conditions.

The above results and the previous discussion on stability of course do not imply that fluctuations cannot have an important effect in systems with periodic boundary conditions. For example, as discussed by Farmer,[7] a system may be stable overall in phase space but have local regions of instability which will result in the amplification of noise. These ideas may be extended to the behavior of systems in ordinary space in the sense that a spatially extended system with periodic boundary conditions may have regions over which a particular state (e.g., a laminar state or a periodic state) of the system is spatially stable (i.e., perturbations are damped as

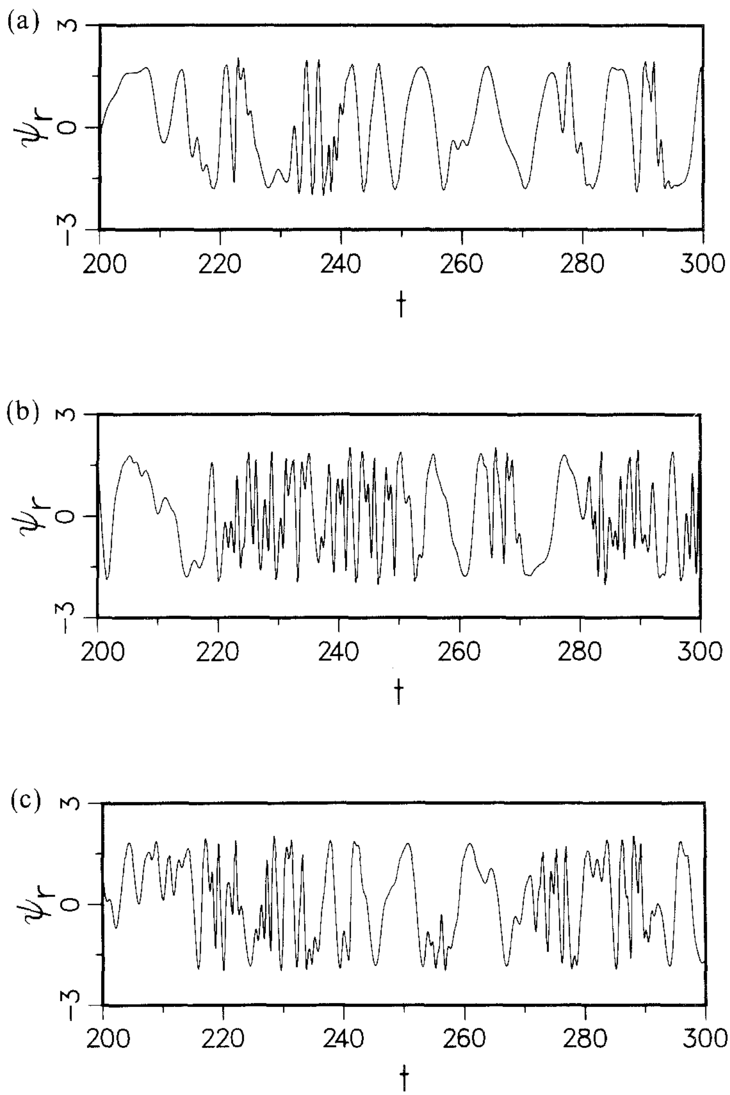

Fig. 8. Plots of ψ_r as a function of t. $a=2$, $v=6$, $b_r=2.8$, $b_i=-1$, $c_r=0.5$, and $c_i=1$. (a) $x=150$, $r=10^{-9}$; (b) $x=210$, $r=10^{-9}$; (c) $x=133.5$, $r=10^{-6}$. High and low frequencies are seen to be interspersed. Also, as a result of the nonlinearity, the high frequencies are much larger than the frequencies in the linear region (compare with Figs. 4b and 4c noting that the time scales differ).

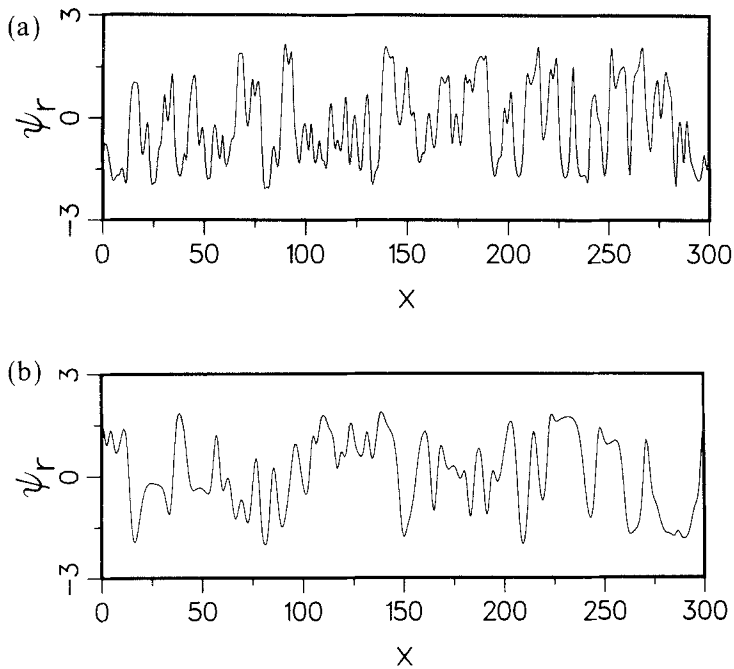

Fig. 9. Plot of ψ_r as a function of x for periodic boundary conditions. $a=2$, $v=6$, $b_i=-1$, $c_r=0.5$ and $c_i=1$, $r=10^{-9}$. (a) $b_r=1$, (b) $b_r=2.8$. Here the overall macroscopic behavior is insensitive to low levels of noise.

they move spatially) and regions over which it is spatially unstable (i.e., perturbations grow as they move spatially). If in the regions where perturbations are damped the fluctuations are sufficiently damped to bring them back below the noise level, noise will be amplified in the spatially unstable regions. For example in the Ginzburg–Landau equation the parameters could be a function of space. In one region the parameters could be such that perturbations grow as they move spatially and in another region they could be such that perturbations are damped.

5. SOME ANALYTICAL RESULTS

We now get an estimate for the spatial growth factor of the fluctuations in the linear region and an estimate for the wave number and frequency of the waves that are selectively amplified. A solution of the linear Eq. (3) is

$$\psi(x, t) = e^{(a-\beta v+\beta^2 b)t} e^{\beta x} \tag{17}$$

Since $|\psi(x, t)|$ is not growing or decaying in time at a given point x after transients have settled down,

$$Re[a - \beta v + \beta^2 b] = 0 \tag{18}$$

This gives us a relationship between β_r and β_i. We first note that only those wave numbers β_i such that $|\beta_i| < (a_r/b_r)^{1/2}$ will be amplified (i.e., $\beta_r > 0$). The wave number that is amplified the most is obtained from $\partial\beta_r/\partial\beta_i = 0$ giving $b_i\beta_r + b_r\beta_i = 0$. Combining this with Eq. (18) gives for the estimated growth factor, wave number, and frequency of the spatially growing waves (assuming $v > 0$)

$$\beta_r = \frac{b_r}{2\,|b|^2}\left[v - \left(v^2 - 4a_r\frac{|b|^2}{b_r}\right)^{1/2}\right] \tag{19}$$

$$\beta_i = -\frac{b_i}{2\,|b|^2}\left[v - \left(v^2 - 4a_r\frac{|b|^2}{b_r}\right)^{1/2}\right] \tag{20}$$

$$\omega = -a_i + \beta_i v - (\beta_r^2 - \beta_i^2)\, b_i - 2\beta_r\beta_i b_r \tag{21}$$

The branch that was chosen was that for which β_r has a maximum. The frequencies predicted by Eq. (21) (i.e., $\omega = 2$ and $\omega = 0.7143$) agree well with the frequencies seen in Figs. 4a and 4b. The frequencies predicted by Eq. (21) are in the limit as $x \to \infty$ which accounts for the discrepancies between the actual and predicted frequencies. Also, assuming that the fluctuations may be seen when they are about 0.05 in magnitude, Eq. (19) predicts that they will be seen at about $x = 46$, $x = 41$, and $x = 28$ for Figs. 1, 2, and 3, respectively. These values appear to agree well with the positions of these points seen in the figures.

The full nonlinear Eq. (2) admits an exact solution (the Stokes solution) of the form

$$\psi(x, t) = Ae^{i(kx - \omega t)} \tag{22}$$

where ω and k are real and where ω, k, and $|A|$ are related by

$$\omega = ia + kv - ik^2 b - ic\,|A|^2 \tag{23}$$

Solving for k and $|A|$ in Eq. (23) gives

$$k = \frac{v \pm [v^2 - 4(b_r c_i/c_r - b_i)(a_i - c_i a_r/c_r + \omega)]^{1/2}}{2(b_r c_i/c_r - b_i)} \tag{24}$$

and

$$|A| = \frac{1}{c_r^{1/2}}\,(a_r - k^2 b_r)^{1/2} \tag{25}$$

An important point here is that the wavelength and the amplitude of the pattern formed depend on the frequency of the spatially growing waves, since this is the frequency at which the pattern is driven. Choosing the negative sign in Eq. (24) and using the parameter values from Fig. 2 and a value for ω from Eq. (21) gives $k = -0.3848$ and $|A| = 1.7807$. These values agree well with the wave number and the amplitude of the regular portion of the structure seen in Fig. 2. It is not clear how to generally choose the sign in Eq. (24) although we note that the sign that gives the correct values for k and $|A|$ for the parameter values in this paper is that sign which gives the larger value for $|A|$ (smaller k^2) and that the analysis of Ref. 33 indicates that longer wavelength solutions are more stable.

Using the parameter values and frequencies from Fig. 5 in Eqs. (24) and (25) (with the negative sign) gives $k = -0.29099$ and $|A| = 1.9572$ for $b_r = 1$; and $k = -0.384768$ and $|A| = 1.78071$ for $b_r = 2.8$. These values agree very well with the actual values (from the numerical solutions) of $k = -0.29101$ and $|A| = 1.9569$ for $b_r = 1$; and $k = -0.384770$ and $|A| = 1.78068$ for $b_r = 2.8$. Also we find that increasing the accuracy of the numerical solutions by decreasing Δx by a factor of 2 and Δt by a factor of 4 results in only a slight change ($<0.015\%$ for $b_r = 1$ and $<0.0015\%$ for $b_r = 2.8$) in k and $|A|$ of Fig. 5. This is a good check on the numerical accuracy of our solutions of the differential equation.

Reference 33 gives conditions to determine whether or not Eq. (22) with periodic boundary conditions is a stable solution of Eq. (2) (actually they treat the case of $v = 0$ but since the boundary conditions are periodic the stability criteria will be independent of v). As noted previously, the sinusoidal patterns of Fig. 5 are spatially unstable. If periodic boundary conditions were instead imposed the structures would be absolutely unstable. Therefore if the stability conditions in Ref. 33 were applied we would expect that they would give the result that the structures are unstable. Putting the parameter values and values for k and $|A|$ from Fig. 5 into the stability criteria [Eqs. (5.6) and (5.7) of Ref. 33] indeed give that the structures are unstable.

Notice that the above equations may be used to determine the likelihood of Eq. (2) producing turbulent behavior for a given set of parameters. Condition (10) may be used to determine whether or not the system is spatially unstable. If it is spatially unstable, Eqs. (19)–(21) may be used to determine the frequency of the spatially growing waves. Then Eqs. (24) and (25) may be used to determine [to within two choices assuming real solutions of Eqs. (24) and (25) exist] the wave number and amplitude of the expected pattern and Ref. 33 may be used to determine whether or not this pattern is stable.[8]

[8] If a is complex, the imaginary part of a may be removed from equation (2) by assuming a solution $\psi(x, t) = \phi(x, t) \exp(ia_i t)$, [19] and Ref. 33 may then be applied.

6. OTHER SYSTEMS

Noise-sustained structure of course is not restricted to equations of the form of Eq. (2). In the following manner we may get conditions for a more general class of equations to be spatially unstable. Given an initial localized perturbation ψ_0 about the equilibrium state $\psi = 0$, the solution of the linear partial differential equation $\partial\psi/\partial t = \sum_{j=0}^{J} a_j(\partial^j\psi/\partial x^j)$ may be written as

$$\psi(x, t) = \int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx} e^{\alpha(k)t} \tag{26}$$

where

$$A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx'\, \psi_0(x')\, e^{-ikx'}$$

Since $\psi_0(x)$ is a localized function of x (i.e., decreases more rapidly than exponentially as $x \to \infty$ and as $x \to -\infty$), $A(k)$ is an analytic function of k in the entire complex plane.[35] Also putting ψ into the above differential equation we see that $\alpha(k)$ is an analytic function of k in the entire complex plane.[9] We assume that $\alpha(k) \to -\infty$ as $k \to \infty$ and as $k \to -\infty$.

The first part of condition (6) [i.e., $\lim_{t \to \infty} |\psi(x, t)| \to 0$ for an arbitrary fixed value of x] is satisfied if $Re[\alpha(k_s)] < 0$, where k_s is the saddle point in the complex k plane. Here we have again used the method of steepest descent.

The second part of condition (6) [i.e., $\lim_{t \to \infty} |\psi(X' + v't, t)| \to \infty$ for some v' and for an arbitrary fixed value of X'] is satisfied if $Re[\alpha(k_m)] > 0$, where k_m is the value of k on the real axis which gives the maximum value of α_r. Although this may be apparent from the fact that the perturbation will grow (for t sufficiently large) if any mode is growing, it may be instructive to derive this result from the method of steepest descent. Transforming to a frame of reference moving at the velocity v' [i.e., substitute x in Eq. (26) with $X' + v't$ where X' is fixed] gives $ikv' + \alpha(k)$ for the coefficient of t. For large t the integral will be dominated by the integrand near the saddle point. Therefore

$$\frac{d}{dk}[ikv' + \alpha(k)] = 0 \tag{27}$$

or since $\alpha(k)$ is analytic and thus satisfies the Cauchy–Riemann equations we have

$$\frac{\partial \alpha_r}{\partial k_r} = 0 \qquad \text{and} \qquad v' = -\frac{\partial \alpha_i}{\partial k_r} \tag{28}$$

[9] For more general cases in which these functions are not analytic in the entire complex plane, care must be taken in properly taking singularities and branch points into account.

Up to this point k_i is still arbitrary and thus k and v' have not yet been determined. To determine k_i we note that the natural velocity of the perturbation corresponds to that v' for which $|\psi(X'+v't, t)|$ grows most rapidly or for which $Re[ikv'+\alpha(k)]$ is maximum. Therefore $(d/dv')\,Re[iv'k(v')+\alpha(k(v'))]=0$ which gives $Re[ik]=0$ or $k_i=0$ where we have used Eq. (27). Thus we have $v'=-d\alpha_i/dk$ (where k is real) which is the familiar expression for the group velocity and $d\alpha_r/dk=0$ (k real) which gives us the desired result that the perturbation will grow if $Re[\alpha(k_m)]>0$.

Therefore the system will be spatially unstable if

$$Re[\alpha(k_s)]<0 \qquad \text{and} \qquad Re[\alpha(k_m)]>0 \tag{29}$$

where k_s is determined from $d\alpha/dk=0$ (k complex) and k_m is determined from $d\alpha_r/dk=0$ (k real). If a derivative gives more than one value of k, that value which gives the maximum α_r is chosen. Note that $Re[\alpha(k_s)]>0$ implies an absolute instability and that $Re[\alpha(k_m)]<0$ implies an absolute stability. For Eq. (3) [where $\alpha(k)=a-ikv-k^2b$], condition (29) reduces to condition (10).

Also, noise-sustained structure will undoubtedly occur in solutions of the Navier–Stokes equations. In fact, for plane Poiseuille flow, the corresponding equation (3) satisfies condition (10), since, as obtained from Ref. 14, $a_r=0.17(10^{-5})(R-R_c)$ where R_c is the critical Reynolds number, $b_r=0.183$, $b_i=0.070$, and $v=0.384$ giving $2\,|b|\,(a_r/b_r)^{1/2}=1.19(10^{-3})(R-R_c)^{1/2}$ which is less than v for the range of R for which the amplitude equation is valid. Unfortunately we cannot compare our numerical solutions with the full nonlinear amplitude equation [i.e., Eq. (2)] derived in Ref. 14 since $c_r<0$ and the amplitude does not saturate.

However, for wind-induced water waves[15] $c_r>0$ for some of the cases considered in that reference and therefore the amplitude does saturate for those cases. A difficulty in numerically solving the amplitude equation with the parameter values for that system is the slow growth rate of the fluctuations. Therefore this problem will await future study and here we just note the parameter values given in that paper satisfy condition (10) by a good margin and thus that system will exhibit noise-sustained structure.

In addition, Refs. 21–23 develop formalism to determine whether a plasma or fluid system will be spatially (i.e., convectively) unstable. Thus the formalism developed in these references may be applied to plasma and fluid systems to indicate the occurrence of noise-sustained structure.

7. CONCLUSIONS

Solutions of the generalized time-dependent Ginzburg–Landau equation in the presence of low-level external noise were studied. It was found that numerical solutions of this equation in the *stationary* frame of reference and with a *nonzero* group velocity that is greater than a critical value exhibits a selective spatial amplification of external noise resulting in spatially growing waves. These waves in turn result in the formation of a dynamic structure which is thus sustained by the presence of the noise. Criteria for the formation of spatially growing waves were given. The *microscopic* noise was found to play a very important role in the *macroscopic* dynamics of the system. One striking feature was the existence of intermittent turbulence similar to that occurring in some fluid systems. A mechanism which may be responsible for the intermittent turbulence occurring in some fluid systems was suggested. From the preceding discussions it is clear that if this mechanism is responsible for the temporal intermittency seen in some systems it will be those systems that exhibit a laminar region followed spatially by a turbulent region. It would be interesting to numerically solve the Navier–Stokes equations in the presence of external noise for a system such as fluid flow over a flat plate to ascertain whether this intermittency mechanism is responsible for the intermittency experimentally seen in that system.

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